Math 53: Multivariable Calculus **Sections 201, 204**

Worksheet answers for 2021-10-20

If you would like clarification on any problems, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to conceptual questions

Question 1.

(a) The region R of integration in the xy-plane is

$$1 \le x \le 4,\ -4 \le y \le 4.$$

The preimage in the uv-plane is

$$1 \le u^2 \le 4,\ -4 \le 2v \le 4$$

i.e. $1 \le |u| \le 2,\ -2 \le v \le 2$. This region consists of two rectangles. The transformation carries each rectangle onto R, so we ought to just take one of the two rectangles as S (so that $S \to R$ is one-to-one). If we take $1 \le u \le 2,\ -2 \le v \le 2$, the integral ends up being

$$\int_1^2 \int_{-2}^2 f(u^2, 2v) 4u \, dv \, du.$$

On the other hand, if we take $-2 \le u \le -1,\ -2 \le v \le 2$, we get

$$\int_{-2}^{-1} \int_{-2}^2 f(u^2, 2v)(-4u) \, dv \, du.$$

Notice that this is because $|4u| = 4u$ on the first choice of S, whereas $|4u| = -4u$ on the second.

(b) The region R of integration in the xy-plane is

$$-4 \le x \le 4,\ 1 \le y \le 4.$$

The stated transformation does not map fully onto R, because it misses everything to the left of the y-axis! This is because $x = u^2 \ge 0$ always. It is impossible to find any region S such that this transformation carries $S \to R$ in a one-to-one and onto fashion (which is necessary for change of variables).

Question 2.

(a) The Jacobian determinant has absolute value 3. The transformation T is one-to-one (since you can solve uniquely for u, v in terms of x, y), so the image of S will have triple the area of S, which is to say that it will have area 12.

(b) The Jacobian determinant is e^u. This depends on u, so the area of R depends on the actual choice of S, not just the area of S.

(c) The Jacobian determinant is 0. The image R will have 0 area.

(d) The Jacobian determinant is 2. However, the transformation T is not one-to-one (in fact it is two-to-one). As such, the area of R could be anything between 4 and 8 inclusive, and more information about S is needed to know the exact value.

Question 3.

(a) Since the transformation T is one-to-one and onto (the onto part is now important, although it wasn't in 2a) the area of S will be $4/3$.

(b) Not enough information, for the same reason as in the preceding problem.

(c) Not enough information. (You would have to know the length of $R \cap C$, where C is the unit circle.)

(d) The transformation T is two-to-one and onto, which means that the preimage of R can be split into two regions, each with area $4/2$ (as the Jacobian determinant is 2). So the total area of the preimage will be 4.

Answers to computations

Problem 1. The integrand strongly suggests the change of variables $u = y - x, v = y + x$. Solving for x, y in terms of u, v we get the transformation

$$x = \frac{1}{2}(v - u),\ y = \frac{1}{2}(u + v)$$

from the uv-plane to the xy-plane. The trapezoidal region is described by

$$x \geq 0,\ y \geq 0,\ 1 \leq x + y \leq 2$$

and thus its preimage in the uv-plane is

$$v - u \geq 0,\ u + v \geq 0,\ 1 \leq v \leq 2.$$

(Since the transformation is one-to-one and onto, this preimage will work as the region S.) The Jacobian determinant works out to $1/2$, so the integrand becomes $\cos(u/v)(1/2)$. This looks a lot easier to integrate with respect to u rather than v, so we settle on the integration order $\mathrm{d}u\,\mathrm{d}v$. The good news is that the region of integration in the uv-plane is conducive to this order as well! Our integral is

$$\int_1^2 \int_{-v}^{v} \frac{1}{2} \cos\left(\frac{u}{v}\right) \mathrm{d}u\,\mathrm{d}v = \boxed{\frac{3}{2}\sin 1}.$$

Problem 2. The region of integration is a rotated square. (It is helpful to think about the given inequality $|x| + |y| \leq 1$ in each of the four quadrants separately.) The entire region is equivalently given by the inequalities

$$-1 \leq y - x \leq 1,\ -1 \leq x + y \leq 1.$$

Looking at this, we see that the change of variables $u = y - x, v = y + x$ should be helpful (coincidentally the same change of variables as in the preceding problem—that was not intentional). Our integral becomes

$$\int_{-1}^{1} \int_{-1}^{1} \frac{1}{2} e^v\,\mathrm{d}v\,\mathrm{d}u = \boxed{e - e^{-1}}.$$